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ON THE METHODOLOGY OF TEACHING INEQUALITIES AND THEIR PROOF BY DIFFERENT METHODS

One of the fundamental concepts in a school mathematic course is inequality. Inequalities are also widely used in other areas of mathematics. Finding the domain, the set of values, increasing (decreasing) intervals of the function, solving some equations, proof issues and other cases are indicators of the solution of inequalities. All this suggests that it is important to teach the concept of inequality in the school mathematics. However, to prove some problems related to inequalities (linear, quadratic, an equation that an unknown includes into modulus sign, fractional rational, irrational, top, logarithmic, trigonometric and system inequalities) by different methods encourages to study the theory of inequalities in depth. Therefore, solving problems related to the proof of inequalities not only helps the deeper and more conscious understanding of the theoretical material, but also teaches students the application of this material in the study of other concepts. In the development of students' logical judgment, besides studies related to solve the inequalities by graphing or interval notation, the solution of other studies on proof is also of great importance. In general, each proof issue should serve a specific purpose as well as a general purpose and it is an important requirement for the proof of inequalities in mathematics. In order to simplify the proof of inequalities, it is suggested to solve various inequalities by the same method in the article. The basis of this method is solving equations and finding the value of an expression with a variable. One of the advantages of the proposed method is that there is no need to learn multiple rules for solving inequalities, and any arbitrary inequalities are also proved by the same algorithm. On the other hand, concepts playing an important role in school mathematics education such as arithmetic and geometric mean theorems, Cauchy's inequality, proof using the mathematical induction method, different ways of proving inequalities by creating the difference between the sides of inequalities have been widely interpreted.

Key words: *inequality, equality, method, number, proof, arithmetic mean, geometric mean, induction.*

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З МЕТОДИКИ НАВЧАННЯ НЕРІВНОСТЕЙ ТА ЇХ ДОВЕДЕННЯ РІЗНИМИ МЕТОДАМИ

Одним із фундаментальних понять шкільного курсу математики є нерівність. Нерівності також широко використовуються в інших областях математики. Показниками розв'язування нерівностей є знаходження області визначення, множини значень, інтервалів зростання (спадання) функції, розв'язування деяких рівнянь, задач на доведення та інші випадки. Усе це говорить про те, що важливо викладати концепцію нерівності в шкільній математиці. Однак доведення деяких задач, пов'язаних з нерівностями (лінійних, квадратних, рівнянь, що невідоме включає знак модуля, дробових раціональних, ірраціональних, вершинних, логарифмічних, тригонометричних і системних нерівностей) різними методами спонукає до поглибленого вивчення теорії нерівностей. Тому розв'язування задач, пов'язаних із доведенням нерівностей, сприяє не тільки глибшому й усвідомленому засвоєнню теоретичного матеріалу, а й вчить учнів застосовувати цей матеріал при вивченні інших понять. У розвитку логічного судження учнів, крім завдань, пов'язаних із розв'язуванням нерівностей за допомогою графіків або інтервальних записів, велике значення має розв'язування інших завдань на доведення. Загалом, кожне завдання з доведенням має служити певній меті, а також загальній меті, і це є важливою вимогою для доведення нерівностей у математиці. Для спрощення доведення нерівностей у статті пропонується розв'язувати різні нерівності одним і тим же методом. В основі цього методу лежить розв'язування рівнянь і знаходження значення виразу зі змінною. Однією з переваг запропонованого методу є те, що для розв'язування нерівностей не потрібно вивчати кілька правил, а будь-які довільні нерівності також доводяться за одним алгоритмом. З іншого *боку, поняття, які відіграють важливу роль у шкільній математичній освіті, такі як теореми про арифметичне та середнє геометричне, нерівність Коші, доведення за допомогою методу математичної індукції, різні способи доведення нерівностей шляхом створення різниці між сторонами нерівностей, отримали широке тлумачення.*

Ключові слова: *нерівність, рівність, спосіб, число, доведення, середнє арифметичне, середнє геометричне, індукція.*

Introduction. The place and role of the inequality in school mathematics course is very crucial. This concept is taught from the lower grades and takes a complex form in the upper grades. From this point of view, the place and role of the concept of inequality in the subject curriculum is clearly defined. One of the main goals is to introduce and consciously educate the scientific and technical achievements of the XXI century to the young generation taking into account the current development trends in the world and Azerbaijan. In order to do this successfully, it is necessary to develop creative and logical thinking based on the rules of didactics of students during the teaching process of mathematics.

The concept of inequality taught in school mathematics is a part of sub-standards of teaching mathematics as a whole, and occupies an important place in solving problems such as transformation of equations, functions, mathematical equations, mathematical expressions. Therefore, the main requirement for teaching the inequality is to determine the method of comprehending based on the internal idea in terms of deep understanding of other concepts in teaching mathematics.

A number of research works have been conducted by experts on teaching inequality in mathematics. Some of them are aimed at the development of general intellectual levels of students, while others are scientifically researched. Quite a few scientific and methodological works of foreign and local experts have been published on teaching the concept of inequality. We can mention the following researchers and mathematicians regarding the proof of some important inequalities: Adigezalov A.S. (Adıgözəlov, 2012), Mammadov A.H, Şukurov R.Y. (Məmmədov, Şükürov, 2011), Yaqubov M. (Yaqubov, 2010), Enderton P.N. (Enderton, 2016), Wen yao Xiong (Wen yao Xiong, 2016), Shahbazov A., Nasirov N.B., Rzayev M.T., Aliyeva T.M. and others.

It is known from the conducted researchs, methodological literatures and textbooks that it is very difficult to teach students the properties of numerical inequalities, ways of proving important inequalities without using the research method and to achieve high results in this direction.

Thus, it is clear from the above mentioned that the problem investigated in this article attracts attention with its topicality.

Currency rate. By teaching the concept of inequality in school mathematics, the formation of the student's personality is achieved, and this process can be associated with the following:

The inequality is one of the important concepts in mathematics, and it is possible to realize intra-disciplinary relations through this concept;

- During the teaching of mathematics, the proof of important properties and theorems related to inequalities is an significant tool to the development of logical thinking for students;
- The proof of simple or complex inequalities plays a crucial role in the formation of students' cognitive activity;
- Solving important inequalities is closely related to intra- and interdisciplinary relationships.;
- Vital components of the inequality are applied in every field of practical activity.;
- In the teaching of other concepts in mathematics, the inequality is considered as one of the main components.

Comprehensive training of students in school mathematics is directly related to the role of future specialists. On the other hand, the methods applied in the training process and the establishment of a new methodical system have also contributions in this direction (Solving inequalities).

Goal and tasks. The development of a new methodical system based on examples of broaden the horizons of scientific outlook and creativity of students through the proof of theorems related to inequalities in mathematics is the main goal of the article.

Tasks:

1) To investigate scientifically and methodically the application of theorems and proofs related to inequalities in mathematics teaching, to practical studies.

2) To identify the research abilities of students using the proof of the Cauchy-Bunyakovsky inequality during the application of practical studies.

Methods – in the article, the following methods were used from the point of view of improving the quality of the educational process.

– application of ICT and modern learning technologies in the proof of some important inequalities.

– to show ways of solving examples of proving inequalities in order to develop logical thinking ability of students by using group, pair, individual or collective work methods and appropriate training forms.

Main part.

Concepts as "Greatness" and "smallness" belonging to the true values of quantities.

"Greatness" and "smallness" are concepts related to the true values of quantities and are applied to many areas of modern mathematics. The concept of inequality is deeply embedded in school mathematics and is gradually developed in all classes of secondary school starting from the first grade. This topic is given enough space in the new subject curriculum (Yaqubov, 2010).

Inequalities is a broader concept than equality, teaching it in school helps students develop logical thinking. It is clear that one of the following three relations is true between two real numbers a and b:

1) $\alpha > b$; 2) $\alpha < b$; 3) $\alpha = b$.

According to the definition of inequality, the difference $a - b$ must be $a>b$ if the result is a positive number, and $a < b$ if it is a negative number.

It is clear from the definition that the inequalities $a > b$ and $a - b > 0$ are equally strong.

If one of them is true, the other is also true, and vice versa, if one is not true, the other is not true. The same words can be said for expressions $a < b$ and $a - b < 0$. The properties of the inequality are proved based on the expressions $a - b > 0$ and $a - b < 0$.

It is known that the solution of inequalities with one or more literal quantities is in one of the following cases:

1) Solving inequalities means finding values of a literal quantity (or quantities) such that the inequality is true.

For example: when solving the inequality $\log_3 \log_2 \frac{6x}{3x-8} > 0$ we first convert the given inequality into the form $\log_2 \frac{6x}{3x-8} > 1$, then the form $\frac{6x}{3x-8} > 2$, then we differentiate between the sides and get $\frac{16}{3x-8} > 0$ and from here, we get x>2, (Elementary Mathematics, 2015).

Thus, the given inequality is true in the set of values of x greater than 2 (Elementary Mathematics, 2015).

2) Prove that the inequality is true for all specified values of the quantities involved.

For example: while proving the inequality $a^2 + b^2 + c^2 > ab + ac + bc$ at arbitrary true values of *a, b* and *c,* firstly we note the expressions $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$ from $(a - b)^2 \geq 0$, $(b - c)^2 \geq 0$ və$(b - a)^2 \geq 0$ inequities that is clear for us, then, based on the known property, we add them side by side and divide all terms of the result by 2. Thus, we get the inequality $a^2 + b^2 + c^2 > ab + ac + bc$ required proof.

There is a certain algorithm to solve the unknown inequality. However, there is no general method for proving inequalities. Each inequality has special methods of proof depending on the given condition and its requirement (Elementary Mathematics, 2015).

We aim to show some of those methods in this article.

Proof of inequalities based on the theorem on arithmetic and geometric mean.

a) First, we write Cauchy's inequality about

$$a_1 + a_2 + a_3 + \cdots + a_n \geq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n} \quad (1)$$

the arithmetic and geometric mean value for an example solution, then we carry out appropriate transformations on that expression and get the required result.

Example 1. Prove that the following relation is true in a triangle.

$$abc \geq (a + b - c)(a + c - b)(b + c - a)$$

Proof. Write the inequalities about the arithmetic and geometric mean:

$$\frac{c+a}{2} \geq \sqrt{ca}, \frac{a+b}{2} \geq \sqrt{ab} \text{ və ya } (b-c)^2 > 0;$$

$$(c-a)^2 \geq 0;\ (a-b)^2 \geq 0$$

Now, show these inequalities as follows.

$$a^2 \geq a^2 - (b - c)^2;\ b^2 \geq b^2 - (c - a)^2;\ c^2 \geq c^2 - (a - b)^2.$$

If we multiply these expressions side by side and perform the following transformations on the right side:

$$a^2b^2c^2 \geq (a + b - c)(a - b + c)(b + c - a) =$$
$$(a + b - c)^2 (a + c - b)^2 (b + c - a)^2$$

Here, we get $abc \geq (a + b - c)(a - b + c)(b + c - a)$.

In some cases, first, we perform certain transformations on the given inequality, then we use the propositions about arithmetic and geometric values and prove the correctness of the required inequality.

Example 2. Prove that the following relation is true for a rectangle.

$$a^2 + b^2 \geq 2S$$

(*a and b are adjacent sides of a rectangle and S is its area*).

Proof. If we apply Cauchy's inequality to the sides of the rectangle:

$$a^2 + b^2 + a^2 + b^2 \geq 4\sqrt[4]{a^2 \cdot b^2 \cdot a^2 \cdot b^2} = 4ab$$

Here we get $a^2 + b^2 \geq 2S$.

If the rectangle is a square, the equation is obtained.

Example 3. If the perimeter of a quadrilateral with sides a, b, c, d is 2 units, prove the truth of the following relation.

$$(2 - a)(2 - b)(2 - c)(2 - d) \geq 16(a + b)(c + d)\sqrt{abcd}$$

Proof. According to condition mentioned above $a + b + c + d = 2$.

Here we get:

$$2 - a = b + c + d = (c + d) + b$$
$$2 - b = c + d + a = (c + d) + a$$
$$2 - c = b + d + a = (a + b) + d$$
$$2 - d = a + b + c = (a + b) + c$$

Apply the Cauchy inequalities about the arithmetic and geometric mean of two numbers to each of these expressions.

$$2-a \geq 2\sqrt{(c+d)\cdot b};\ 2-b \geq 2\sqrt{(c+d)\cdot a};$$
$$2-c \geq 2\sqrt{(a+b)\cdot d};\ 2-d \geq 2\sqrt{(a+b)\cdot c}.$$

If we multiply the obtained inequalities side by side.

We get the result $(2-a)(2-b)(2-c)(2-d) \geq 16(a+b)(c+d)\sqrt{abcd}$.

Example 4. Prove the inequality: $\frac{a^2+4}{\sqrt{a^2+3}} \geq 2$.

Proof. Write the left side of the expression like this $\sqrt{a^2+3}+\frac{1}{\sqrt{a^2+3}}$ and apply the inequality $\frac{x+y}{2} \geq \sqrt{xy}$ to that: $\sqrt{a^2+3}+\frac{1}{\sqrt{a^2+3}} \geq 2\sqrt{\sqrt{a^2+3}\cdot\frac{1}{\sqrt{a^2+3}}}$ or $\frac{a^2+3+1}{\sqrt{a^2+3}} \geq 2$.

Here, we get $\frac{a^2+4}{\sqrt{a^2+3}} \geq 2$.

Proof of inequalities by differentiating between the sides. To prove inequalities in this way, we first differentiate between the sides. For this reason, we transfer the right side to the left side, then perform transformations on the received expression and bring it to a simple form, and make a judgment on this last expression and determine that it is greater than and less than zero (Solving linear inequalities). As a result of this, we define whether the inequality requiring to be proved is true or not.

We will explain what we are saying with the following examples.

Example 1. If $a > 0$ və $b > 0$, prove the inequality $a^3 + b^3 \geq a^2b + ab^2$.

Proof. First, transfer the right side to the left side and perform transformations on the obtained difference.

$$a^3 + b^3 - a^2b - ab^2 \geq 0$$
$$a^2(a-b) - b^2(a-b) \geq 0$$
$$(a-b)(a^2-bv) \geq 0$$
$$(a-b)^2(a+b) \geq 0$$

It is obvious from the last inequality that the first term is not less than zero. The second one is always greater than zero within the given condition ($a > 0$; $b > 0$). Therefore, their product is not less than zero.

Thus the truth of last inequality shows that $a^3 + b^3 \geq av\,b + ab^2$ is true.

Example 2. Prove the following inequality for an arbitrary cone with volume V and lateral surface area S.

$$\left(\frac{6V}{\pi}\right)^2 \leq \left(\frac{2S}{\pi\sqrt{3}}\right)^3$$

Proof. If we denote the radius and the slant height of the seat of the cone by R and l respectively, its height is $\sqrt{l^2-R^2}$, then the volume and surface of the cone respectively are $V=\frac{1}{3}\pi R^2\sqrt{l^2-R^2}$ və $S=\pi Rl$.

If we write these values found in the inequality required to be proved, then divide both sides of it by $4R^3l^3$ and perform the following transformations on the obtained expression:

$$4(R^4l^2-R^6) \leq \frac{8}{(\sqrt{3})^3}R^3l^3.$$

Here, it can be $\frac{R^4l^2-R^6}{R^3l^3} \leq \frac{2}{(\sqrt{3})^3}$ or $\frac{R}{l}-\frac{R^3}{l^3} \leq \frac{2}{(\sqrt{3})^3}$.

Now create the difference between the limits of the last inequality as follows:

$$\left(\frac{R}{l}\right)^3-\left(\frac{1}{\sqrt{3}}\right)^3-\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right) \geq 0$$
$$\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\left[\left(\left(\frac{R}{l}\right)^2+\frac{R}{l}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}\right)\right]-\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right) \geq 0$$
$$\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\left[\left(\frac{R}{l}\right)^2+\frac{R}{l}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}-1\right] \geq 0 \qquad (1)$$

Write the inequality as follows and perform the indicated transformations on it:

$$\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\left[\left(\frac{R}{l}\right)^2+\frac{R}{l}\cdot\frac{1}{\sqrt{3}}-\frac{2}{3}\right] \geq 0$$
$$\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\left[\left(\frac{R}{l}\right)^2-\frac{R}{l}\cdot\frac{1}{\sqrt{3}}+\frac{R}{l}\cdot\frac{2}{\sqrt{3}}-\frac{2}{3}\right] \geq 0$$
$$\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\left[\frac{R}{l}\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)\right] \geq 0$$

Here, we get $\left(\frac{R}{l}-\frac{1}{\sqrt{3}}\right)^2\left(\frac{R}{l}+\frac{2}{\sqrt{3}}\right) \geq 0$.

From this it is obvious that the first factor is always positive. And in the second factor, since $\frac{R}{l} < 1$ it is also always positive.

Thus, the inequality is proved to be true.

Example 3. If $ab \geq 0$ $a^2 - b^2 \geq (a-b)^4$ prove the truth of inequality.

Proof. Firstly, make the following difference between the terms of the inequality, and then perform the indicated transformations on it. It is obvious that, it is enough to prove the inequality $(a^2-b^2)^2 - a - b)^4 \geq 0$ is sufficient instead of the required inequality.

$$[(a^2-b^2)-(a-b)^4][(a^2-b^2)^2+(a-b)^2] =$$
$$= (a^2-b^2-a^2+2ab-b^2) \times (a^2-b^2+a^2-2ab+$$
$$+b^2) = (2ab-2b^2)(2a^2-2ab) = 4ab(a-b) \times$$
$$\times (a-b) = 4ab(a-b)^2.$$

So, we get $ab(a-b)^2 \geq 0$. According to condition, the first factor is $ab > 0$. Second factor is always positive. If $a = b$, result is zero.

Thus, we get $(a^2-b^2)^2 \geq (a-b)^4$.

Example 4. If $a > b > c$ or $c > a > b$ or $b > c > a$, prove that $a^2b + b^2c + c^2a > a^2c + c^2b + b^2 a$ is true.

Proof. First, transfer the right side of the inequality required to be proved to the left and perform the indicated transformations on it. It is obvious that, it is enough to prove the inequality $a^2b + b^2c + c^2a - a^2c - c^2b - b^2a > 0$ is sufficient instead of the required inequality.

$a^2b + b^2c + c^2a - a^2c - c^2b - b^2a = abc - b^2a - a^2c +$
$+ a^2b - c^2b + b^2c + c^2a - abc = a(bc - b^2 - ac + ab) -$
$- c(ab - b^2 - ac + ab) = a[b(c - b) - a(c - b)] -$
$- c[(b(c - b) - a(c - b)] = a(c - b)(b - a) - c(c - b)$
$(b - a) = (c - b)(b - a)(a - c) > 0$

1) If $a > b > c$, $c - b < 0$; $b - a < 0$ is true; $a - c > 0$, and $(c - b)(b - a)(a - c) > 0$ is also true.

2) If $c > a > b$, $c - a > 0$; $b - a < 0$; $a - c < 0$ is true, and $(c - b)(b - a)(a - c) > 0$ is also true.

3) If $b > c > a$, $c - b < 0$; $b - a > 0$ is true; $a - c < 0$, $(c - b)(b - a)(a - c) > 0$ is also true.

Note. If $a < b < c$ or $c < a < b$ or $b < c < a$, it can be proved by the rule above mentioned that $a^2b + b^2c + c^2a < a^2c + c^2b + b^2 a$ is true.

Proof of inequality by comparing its numbers with other ones. In order to prove the inequality, first, each number in the inequality is compared with the same or other numbers. Then transformations performed on the expressions obtained as a result of the comparison according to the given example.

Example 1. Prove the following inequality when n is a positive integer:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2}$$

Proof. Compare each number of the given inequality to the $\frac{1}{2n}$:

$$\frac{1}{2n-1} > \frac{1}{2n}; \ldots; \frac{1}{n+2} > \frac{1}{2n}; \ \frac{1}{n+1} > \frac{1}{2n}$$

if we sum each number of this inequality side by side:

We get $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} =$

$$= \frac{1+1+1+\cdots+1}{2n} = \frac{n}{2n} = \frac{1}{2}.$$

Example 2. Prove the truth of following inequality:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{97}{98} \cdot \frac{99}{100} < \frac{1}{10}$$

Proof: Compare each term in the left side with the numbers that are greater than it:

$$\frac{1}{2} < \frac{2}{3}$$
$$\frac{3}{4} < \frac{4}{5}$$
$$\frac{5}{6} < \frac{6}{7}$$
$$\ldots \ldots \ldots$$
$$\frac{97}{98} < \frac{98}{99}$$
$$\frac{99}{100} < \frac{100}{101}$$

If we multiply the inequalities side by side:

If $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{97}{98} \cdot \frac{99}{100} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{98}{99} \cdot \frac{100}{101} = \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{98}{97} \cdot \frac{100}{99} \cdot \frac{1}{101} = \frac{1}{\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{97}{98}\frac{99}{100}} \cdot \frac{1}{101}$, the result is

$$\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{97}{98} \cdot \frac{99}{100}\right)^2 < \frac{1}{100}.$$

Example 3. Prove the following inequalities:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2 .$$

Proof. If $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$, the result is $\frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2\sqrt{n}$.

Compare each number in the left side of the given inequality according to the last expression as follows:

$$1 > 2\sqrt{2} - 2$$
$$\frac{1}{\sqrt{2}} > 2\sqrt{3} - 2\sqrt{2}$$
$$\frac{1}{\sqrt{3}} > 2\sqrt{4} - 2\sqrt{3}$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\frac{1}{\sqrt{n-1}} > 2\sqrt{n} - 2\sqrt{n-1}$$
$$\frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2\sqrt{n}$$

If we sum the inequalities side by side: the result is

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Example 4. Prove that the following inequality is true when n is an arbitrary number.

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}$$

Proof. Write the following inequalities known to be true:

$$\frac{1}{2^2} < \frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}$$
$$\frac{1}{3^2} < \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\frac{1}{n^2} < \frac{1}{(n-1) \cdot n} = \frac{1}{n-1} - \frac{1}{n}$$

Sum them side by side:

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} = 1 - \frac{1}{n} = \frac{n-1}{n}.$$

So, we get $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \leq \frac{n-1}{n}.$

Proof of inequalities using the method of mathematical induction

Inequalities related to natural numbers are mainly proved by this method.

Misal 1. Prove the truth of the following inequality when n is an arbitrary natural number greater than unity.

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}.$$

Proof. 1) If $n + 2$, by indicating $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}>\sqrt{2}$, the result can be $\frac{\sqrt{2}+1}{\sqrt{2}}>\sqrt{2}$ or $\sqrt{2}+1>2$.

Now, assume that, inequality is true if $n + k$.

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}>\sqrt{k}.$$

Prove that, same inequality is true when $n + k + 1$.

So $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k}+\frac{1}{\sqrt{k+1}}$.

Perform the following transformations on the right side of the last expression.

$$\sqrt{k}+\frac{1}{\sqrt{k+1}}=\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}=\frac{\sqrt{k+1}(\sqrt{k(k+1)}+1)}{k+1}=$$

$$=\sqrt{k+1}\frac{\sqrt{k(k+1)}+1}{k+1}.$$

According to condition $k > 1$. So, it should be $\sqrt{k(k+1)}>k$.

So the last expression is greater than $\sqrt{k+1}$. Thus, the truth of the inequality for $n + k + 1$ is revealed.

Example 2. If $n \geq 2$, $2!\cdot 4!\cdot\ldots\cdot(2n)! > [(n+1)!]^n$ prove the truth of inequality.

Proof. if $n + 2$ the inequality becomes:

$2!\cdot 4!(3!)^2$ or $1\cdot 2\cdot 1\cdot 2\cdot 3\cdot 4 > (1\cdot 2\cdot 3)^2$; $48 > 36$

So, the inequality is true, when $n + 2$.

Assume that, the inequality is true, when $n + k - 1$. So

$$2!\cdot 4!\cdot\ldots\cdot(2k-2)! > [(k!)]^{(k-1)} \qquad (1)$$

Now, prove that the inequality is true, if $n + k$. If we multiply each side of inequality $2!\cdot 4!\cdot\ldots\cdot(2k)! > [(k+1)!]^k$ (1) by $(2k)!$, we get: $2!\cdot 4!\cdot\ldots\cdot(2k-2)!\cdot(2k)! > (2k)![(k)!]^{(k-1)} = \frac{(2k)![(k)!]^{k-1}}{k!}$.

After dividing, we get the following inequality.

$$2!\cdot 4!\cdot\ldots\cdot(2k-2)!\cdot(2k)! > (2k)\cdot(2k-1)\cdot\ldots\cdot(k+1)[(k)!]^k$$

On the other hand, we know,

$$(2k)\cdot(2k-1)\cdot\ldots\cdot(k+1)\,[(2k)!]^k > (k+1)^k(k!)^k + [(k+1)!]^k$$

We get from last two inequlities,

$$2!\cdot 4!\cdot\ldots\cdot(2k-2)!\cdot(2k)! > [(k+1)!]^k$$

So we established that the inequality is true when n + 2, then we showed that the inequality is also true when we go from n + k – 1 to n + k. Based on the principle of mathematical induction, we get that the inequality is true for arbitrary values of n not smaller than 2.

Conclusion. How the concept of equality is perceived by students should not be limited to the teaching of the topics provided in the program, and knowledge, skills and habits should be formed in students through materials collected from various literatures. Therefore, proving the theorems related to inequalities with different methods develops thinking activity of students. Therefore, the selection and application of special methods related to inequalities is of great importance. Mathematical properties of equations and inequalities taught in school mathematics are designed in such a way that the logical thinking of students develops by the mentioned concepts. Factors such as analysis, synthesis, generalization, and abstraction are evident when students solve tasks related to the proof of inequalities.

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